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ON THE SOLUTION OF CERTAIN BOUNDARY VALUE PROBLEMS OF HEAT CONDUCTION

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TABLE OF CONTENTS

INTRODUCTION

For many years mathematicians have been interested in nonlinear differential equations of mathematical physics. In recent years the interest has heightened because of the nonlinear character of many of the problems encountered in the current attempts at space exploration. Unfortunately, exact solutions of most nonlinear differential equations are difficult, if not impossible, to obtain in closed form. Hence the study of methods for approximate solutions to such problems is of importance.

A class of nonlinear parabolic partial differential equations is discussed in this paper. In particular, a fairly general class of nonlinear heat conduction equations with possibly nonlinear mixed boundary conditions is considered; namely, the system

$$u_{xx} - F(u)u_t = G(u,x,t), \qquad (x,t) \in (0,L)x(0,T),$$

$$(1.1) \quad \lim_{t \to 0} u(x,t) = g(x),$$

$$\frac{\partial u}{\partial x}(0,t) = f_1(u,t), \qquad \frac{\partial u}{\partial x}(L,t) = f_2(u,t).$$

An associated integral equation, constructed from the partial differential equation system 1.1, is presented and a classical method of successive approximations is used to solve the integral equation.

Special emphasis is given to the following heat conduction problems: (1) heat equation with conductivity dependent on the temperature; (2) nonlinear heat flux at the boundary.

Gevrey (10) presented existence theorems for nonlinear parabolic problems as early as 1913. Bernstein (2), in an excellent summary, has presented a collection of existence theorems in partial differential equations. More recently Čžou (5), (6), (7), has considered certain nonlinear problems of the form

$$u_{xx} - A(x,t,u)u_t = B(x,t,u,u_x) \tag{1.2}$$

with mixed boundary conditions using the techniques of Rothe (15) and Schauder (16), (17), (18). Unfortunately such techniques, in general, either do not construct a solution or utilize auxiliary functions which are known to exist but difficult, if not impossible, to exhibit. Chambre (4), using a Laplace transform technique, has discussed the mathematical problem associated with heat conduction in the semi-infinite rod:

$$\begin{aligned} &u_{xx} - u_t = 0 \\ &u(x,0) = u_0 \\ &u_x(0,t) = f(u(0,t),t) \\ &\lim_{x \to \infty} u(x,t) = 0. \end{aligned} \tag{1.3}$$

He has also given numerical results for the important cases in which $f(u(0,t),t) = ku(0,t)$ and $f(u(0,t),t) = k(u^4(0,t) - 1)$.

The method of successive approximation was chosen for three reasons: (1) it constructs an actual solution; (2) it is a simple, straightforward technique; (3) the repetitive nature of the approximation would seem to be natural for high-speed computer calculation.

This method, paradoxically, often poses a serious problem for the electronic computer. As in the present case, the kernel function of the integral equation may have a singularity even though the integral exists. Such a singularity may result in the computer requiring excessively small step sizes. This in turn results in so many computational operations that the cost of computer time is prohibitive and the cumulative truncation error exceeds acceptable bounds. In some cases, however, integration over the singularity may be simplified for computational purposes by a change of variables.

In the following section preliminary results are discussed. In the next section an existence proof and a discussion of the resulting approximation is presented. The final section discusses numerical results for two simple nonlinear systems so that the usefulness of the approximation procedure may be examined further.

PRELIMINARY RESULTS

In this section we discuss the domain of definition of our problem and present several lemmas concerning the properties of the fundamental solution of the adjoint to the heat equation; that is, of the equation $u_{xx} + u_t = 0$.

Let $D(T)$ be an open two-dimensional domain of real variables (x, t) which is bounded by the two lines $t = 0$ and $t = T > 0$ and the lines $x = a$ and $x = b$. Denote by $\overline{D}(T)$ the closure of $D(T)$. Denote by $B(t)$ the set of points

$$\{(x,y): x \in (a,b),\ y = t\}.$$

Without loss of generality we may assume $a = 0$ and $b = L$ where L is fixed, positive, real number (or $L = \infty$). We denote by α the integration or running variable in the x-space coordinate and by β the integration variable in the t coordinate. For convenience in notation we will denote by $[F(\alpha, \beta)]_{(0,L)}$ an expression of the form $F(L, \beta) - F(0, \beta)$.

In the following lemmas let $K(\alpha, \beta;\ x, t)$ be the fundamental solution of the adjoint to the heat equation; that is,

$$K(\alpha, \beta;\ x, t) = [4\pi(t - \beta)]^{-1/2} \exp\left[\frac{-(x - \alpha)^2}{4(t - \beta)}\right], \tag{2.1}$$
$$0 \leq \beta < t.$$

Lemmas 1, 2 and 3 are statements of well-known properties of the fundamental solution $K(\alpha, \beta;\ x, t)$. See, for example, Maple (13), Hadamard (12), Gevrey (10), Dressel (8), (9), and Muntz (14). Bernstein (2) includes a comprehensive list of references in her bibliography.

Lemma 1: Let $H(\alpha, \beta)$ be continuous on $\bar{D}(T)$. Then

$$\lim_{\beta \to t^-} \int_0^L H(\alpha, \beta)\, K(\alpha, \beta;\, x, t)\, d\alpha = H(x, t). \tag{2.2}$$

Lemma 2: Let $H_1(\beta)$ and $H_2(\beta)$ be continuous for $0 \leq \beta \leq T$, let $H(\alpha, \beta)$ be continuous and bounded and let $\bar{H}(\alpha, \beta)$ have bounded continuous derivatives of order two in α and one in β for $0 < \alpha < L$ and $0 < \beta \leq T$.

$$I_1(x, t) = \int_0^t \left[H_1(\beta) K(L, \beta;\, x, t) - H_2(\beta) K(0, \beta;\, x, t) \right] d\beta, \tag{2.3}$$

$$I_2(x, t) = \int_0^t \left[H_1(\beta) K_\alpha(L, \beta; x, t) - H_2(\beta) K_\alpha(0, \beta; x, t) \right] d\beta, \tag{2.4}$$

$$I_3(x, t) = \int_0^t \int_0^L H(\alpha, \beta;\, x, t)\, d\alpha d\beta, \tag{2.5}$$

and

$$I_4(x, t) = \int_0^t \int_0^L \bar{H}(\alpha, \beta)\, K_{\alpha\alpha}(\alpha, \beta;\, x, t)\, d\alpha d\beta \tag{2.6}$$

have the following properties:

(1) I_i, $(i = 1, 2, 3, 4)$, have bounded continuous derivatives of order two in x and order one in t for $0 < x < L$, $0 < t \leq T$.

(2) $\lim\limits_{x \to 0^+} I_i$, $\lim\limits_{x \to L^-} I_i$, $\lim\limits_{x \to 0^+} \frac{\partial}{\partial x} I_i$, and $\lim\limits_{x \to L^-} \frac{\partial}{\partial x} I_i$,

$(i = 1, 2, 3, 4)$, exist and are continuous functions of t, $0 < t \leq T$.

Lemma 2a: Let $H_3(\alpha)$ be continuous for $0 \leq \alpha \leq L$. Then the integral $I_5(x,t) = \int_0^L H_3(\alpha)\ K(\alpha,0;\ x,t)d\alpha$ has properties (1) and (2) of Lemma 2 and $\lim_{t \to 0^+} I(x,t) = H_3(x)$.

Gevrey (10), has shown that if we have the additional conditions that H, H_1, H_2, H_3, and $\bar{H}$ are analytic, then I_i, $i = 1, 2, 3, 4, 5$, are analytic.

Lemma 3 presents an additional property of the fundamental solution $K(\alpha,\beta;\ x,t)$.

Lemma 3: Let $H(\alpha,\beta)$ be continuous and bounded for $0 < \alpha < L$, $0 \leq \beta \leq T$ and let $G(\alpha,\beta)$ be a continuous function of β for $\alpha = 0$ and $\alpha = L$, $0 \leq \beta \leq T$. Then

$$(2.7) \qquad \int_0^t \int_0^L H(\alpha,\beta)\ K(\alpha,\beta;\ x,t)d\alpha\ d\beta \equiv \int_0^t \left[G(\alpha,\beta)\ K(\alpha,\beta;\ x,t) \right]_{(0,L)} d\beta$$

implies that $H(\alpha,\beta) \equiv G(0,\beta) \equiv G(L,\beta) \equiv 0$.

Proof: From Lemma 2 and the hypotheses on $H(\alpha,\beta)$ and $G(\alpha,\beta)$ we see that we may take the Laplace transform of Equation 2.7 to write

$$(2.8)\quad \int_0^{\infty} \exp(-st)\left\{\int_0^t\int_0^L H(\alpha,\beta)\; K(\alpha,\beta;\; x,t)d\alpha\; d\beta\right\} dt$$

$$\equiv \int_0^{\infty} \exp(-st)\left\{\int_0^t \Big[G(\alpha,\beta)\; K(\alpha,\beta;\; x,t)\Big]_{(0,L)} d\beta\right\} dt.$$

By Fubini's Theorem [see, for example, Widder, (19), pp. 25-26, for a general statement of Fubini's Theorem] we may write

$$(2.9)\quad \int_0^L\int_0^{\infty} \exp(-st)\left\{\int_0^t H(\alpha,\beta)\; K(\alpha,\beta;\; x,t)d\beta\right\} dt\; d\alpha$$

$$\equiv \int_0^{\infty} \exp(-st)\left\{\int_0^t \Big[G(\alpha,\beta)\; K(\alpha,\beta;\; x,t)\Big]_{(0,L)} d\beta\right\} dt.$$

Since

$$K(\alpha,\beta;\; x,t) = \left[4\pi(t-\beta)\right]^{-1/2} \exp\left[\frac{-(x-\alpha)^2}{4(t-\beta)}\right]$$

we see that we may use the convolution theorem to write Equation 2.9 in the form

$$(2.10)\quad \int_0^L L\{H(\alpha,t)\}\; s^{-1/2} \exp\left[-|x-\alpha|\; s^{1/2}\right] d\alpha$$

$$= L\{G(L,t)\} \exp\left[-|L-x|\; s^{1/2}\right] s^{-1/2}$$

$$- L\{G(0,t)\} \exp\left[-|x|\; s^{1/2}\right] s^{-1/2},$$

$$s > 0,$$

where we have used the fact that

$$L_t\left\{(\pi t)^{-1/2} \exp\left[\frac{-(x-\alpha)^2}{4t}\right]\right\} = s^{-1/2} \exp\left[-|x-\alpha|\; s^{1/2}\right],$$

$$s > 0.$$

If we write $L\{H(\alpha,t)\} = f(\alpha,s)$, $L\{G(L,t)\} = g_1(s)$, and $L\{G(0,t)\} = g_2(s)$, and multiply both sides by $s^{1/2}$, we may express Equation 2.10 in the form

$$(2.11)\qquad \int_0^L f(\alpha,s)\exp\left[-|x-\alpha|\,s^{1/2}\right]d\alpha = g_1(s)\exp\left[-|L-x|\,s^{1/2}\right] - g_2(s)\exp\left[-|x|\,s^{1/2}\right].$$

For $0 < x < L$ we may write Equation 2.11 in the form

$$(2.12)\qquad g_1(s)\exp\left[-(L-x)s^{1/2}\right] - g_2(s)\exp\left[-xs^{1/2}\right] = \exp\left[-xs^{1/2}\right]\int_0^x f(\alpha,s)\exp\left[\alpha s^{1/2}\right]d\alpha + \exp\left[xs^{1/2}\right]\int_x^L f(\alpha,s)\exp\left[-\alpha s^{1/2}\right]d\alpha.$$

If we now divide both sides of Equation 2.12 by $\exp\left[-xs^{1/2}\right]$ we get

$$(2.13)\qquad g_1(s)\exp\left[-(L-2x)\,s^{1/2}\right] - g_2(s) = \int_0^x f(\alpha,s)\exp\left[\alpha s^{1/2}\right]d\alpha + \exp\left[2xs^{1/2}\right]\int_x^L f(\alpha,s)\exp\left[-\alpha s^{1/2}\right]d\alpha.$$

We now differentiate Equation 2.13 with respect to x to get

$$(2.14) \qquad 2s^{1/2}\, g_1(s)\, \exp[-(L - 2x)s^{1/2}]$$

$$= f(x,s)\, \exp[xs^{1/2}]$$

$$+ 2s^{1/2}\, \exp|2xs^{1/2}| \int_x^L f(\alpha,s)\, \exp[-\alpha s^{1/2}]\, d\alpha$$

$$- \exp[2xs^{1/2}] f(x,s)\, \exp[-xs^{1/2}]$$

$$= 2s^{1/2}\, \exp[2xs^{1/2}] \int_x^L f(\alpha,s)\, \exp[-\alpha s^{1/2}]\, d\alpha.$$

If we divide through by $2s^{1/2} \exp[-(L - 2x)s^{1/2}]$ we get

$$(2.15) \qquad g_1(s) = \exp[Ls^{1/2}] \int_x^L f(\alpha,s)\, \exp[-\alpha s^{1/2}]\, d\alpha.$$

Finally, we differentiate again with respect to x to get

$$(2.16) \qquad 0 = \exp[Ls^{1/2}] f(x,s)\, \exp[-xs^{1/2}].$$

But the exponential terms are not zero. Hence $f(x,s) = 0$. Moreover, by Lerch's theorem, if the Laplace transform of a function is zero, the function is zero. Therefore, $H(\alpha,\beta) \equiv 0$. Further, from Equation 2.16, we see that $g_1(s) = 0$ and, similarly we see that $g_2(s)$ must also be zero. We again apply Lerch's Theorem to show that $G(0,t) = G(L,t) = 0$ and our lemma is proved.

AN EXISTENCE THEOREM AND APPROXIMATION

We consider the following mixed boundary value problem for the nonlinear heat equation:

$$(3.1) \quad N(u) \equiv u_{xx} - F(u)u_t = G(u,x,t), \qquad (x,t) \in D(T),$$

$$\lim_{t \to 0^+} u(x,t) = g(x),$$

$$(3.2) \qquad \frac{\partial u}{\partial x}(0,t) = f_1(u,t),$$

$$\frac{\partial u}{\partial x}(L,t) = f_2(u,t).$$

<u>Theorem</u>: Let

(1) $g(x)$ be continuous on $[0,L]$,

(2) $F(u)$ be of class $C^{(1)}$ and $F(u) > 0$,

(3) $H(u) \equiv \int^u F(u)du$,

(4) H^{-1} have a unique real positive branch,

(5) $G(u,x,t)$ satisfy the Lipschitz condition $|G(u_1,x,t) - G(u_2,x,t)| \leq M_3|u_1 - u_2|$, where M_3 is a positive constant for any fixed $t < T$, and let $G(u,x,t)$ be continuous and bounded for $(x,t) \in D(T)$.

(6) $f_i(u,t)$, $i = 1, 2$, satisfy the Lipschitz condition $|f_i(u_1) - f_i(u_2,t)| \leq M_4|u_1 - u_2|$, where M_4 is a positive constant for any fixed $t < T$, and let $f_i(u,t)$ be continuous in t, $0 \leq t \leq T$.

Then there exists a solution $u(x,t)$ of Equation 3.1 satisfying Equation 3.2.

Proof: The proof is based on the reduction of the system 3.1 and 3.2 to an integral equation. In this proof we use the properties of the fundamental solution $K(\alpha,\beta;\ x,t)$ of the equation $L^*(v) = v_{\alpha\alpha} + v_\beta = 0$. In order to do this, we first develop an expression in $N(u)$ and $L^*(K)$ such that the terms involving the derivatives with respect to β may be put in the form $\frac{\partial}{\partial\beta}\left[h(\alpha,\beta)K(\alpha,\beta;\ x,t)\right]$. Then we integrate the entire expression over $D(y)$, $0 < y < t$, carry out the integration with respect to β of the term $\frac{\partial}{\partial\beta}(hK)$, and apply Lemma 1 to extract the function $h(x,t)$. This results in a nonlinear Volterra integral equation for the determination of $u(x,t)$. We then use the method of successive approximations to solve the resulting integral equation.

We first consider the identity

$$(3.3) \qquad KN(u) - H(u)L^*(K) = KG(u,\alpha,\beta).$$

Since $\frac{\partial}{\partial\beta}H(u) = F(u)u_\beta$, Equation 3.3 may be expressed in the form

$$(3.4) \qquad 0 = K\left[u_{\alpha\alpha} - \frac{\partial}{\partial\beta}H(u)\right] - H(u)\left[K_{\alpha\alpha} + K_\beta\right] - KG(u,\alpha,\beta)$$

$$= Ku_{\alpha\alpha} - H(u)K_{\alpha\alpha} - \frac{\partial}{\partial\beta}\left[H(u)K\right] - KG(u,\alpha,\beta).$$

We integrate Equation 3.4 over $D(y)$ to get

$$(3.5)\qquad 0 = \int_0^y\int_0^L \left[Ku_{\alpha\alpha} - H(u)K_{\alpha\alpha} \right] d\alpha\, d\beta$$

$$- \int_0^y\int_0^L \frac{\partial}{\partial\beta}\left[H(u)K \right] d\alpha\, d\beta - \int_0^y\int_0^L KG(u,\alpha,\beta)\, d\alpha\, d\beta.$$

At the endpoints 0 and L we have the conditions 3.2. Hence we may integrate by parts to express the integral

$$\int_0^y\int_0^L Ku_{\alpha\alpha} d\alpha\, d\beta$$

in the form of a known integral with respect to β only. If we perform this integration, Equation 3.5 becomes

$$(3.6)\qquad 0 = \int_0^y \left[Ku_\alpha \right]_{(0,L)} d\beta - \int_0^y\int_0^L K_\alpha u_\alpha d\alpha\, d\beta$$

$$- \int_0^y\int_0^L H(u)K_{\alpha\alpha} d\alpha\, d\beta - \int_0^y\int_0^L \frac{\partial}{\partial\beta}\left[H(u)K \right] d\alpha\, d\beta$$

$$- \int_0^y\int_0^L KG(u,\alpha,\beta)\, d\alpha\, d\beta.$$

If we now integrate the term $\int_0^y\int_0^L K_\alpha u_\alpha d\alpha\, d\beta$ by parts with respect to α and carry out the integration of the term $\int_0^y\int_0^L \frac{\partial}{\partial\beta}\left[H(u)K \right] d\alpha\, d\beta$ with respect to β, Equation 3.5 becomes

$$(3.7)\qquad 0 = \int_0^y \left[Ku_\alpha \right]_{(0,L)} d\beta - \int_0^y \left[uK_\alpha \right]_{(0,L)} + \int_0^y\int_0^L uK_{\alpha\alpha} d\alpha\, d\beta$$

$$- \int_0^y \int_0^L H(u)K_{\alpha\alpha} d\alpha \, d\beta - \int_0^L KG(u,\alpha,\beta) d\alpha \, d\beta$$

$$- \int_0^L H(u(\alpha,y))K(\alpha,y;\ x,t) d\alpha$$

$$+ \int_0^L H(u(\alpha,0))K(\alpha,0;\ x,t) d\alpha.$$

We now take the limit of Equation 3.7 as $y \to t^-$. By Lemma 1 we have

$$\text{(3.8)} \quad \lim_{y \to t^-} \int_0^L H(u(\alpha,y))K(\alpha,y;\ x,t) d\alpha = H(u(x,t)).$$

By Lemma 2 we know that the other integrals exist in the limit. Hence if we collect and rearrange terms in Equation 3.7 we get

$$\text{(3.9)} \quad H(u(x,t)) = \int_0^L H(u(\alpha,0))K(\alpha,0;\ x,t) d\alpha$$

$$+ \int_0^t \left[Ku_\alpha - uK_\alpha \right]_{(0,L)} d\beta$$

$$- \int_0^t \int_0^L \{K_{\alpha\alpha}[H(u) - u] + KG(u,\alpha,\beta)\} d\alpha \, d\beta.$$

If we now substitute the boundary conditions 3.2 into Equation 3.9 we get

$$\text{(3.10)} \quad H(u(x,t)) = \int_0^L H(g(\alpha))K(\alpha,0;\ x,t) d\alpha$$

$$+ \int_0^t \left[Kf(u) - uK_\alpha \right]_{(0,L)} d\beta$$

$$- \int_0^t \int_0^L \{K_{\alpha\alpha}[H(u) - u] + KG(u,\alpha,\beta)\} d\alpha \, d\beta$$

where $[Kf(u)]_{(0,L)} \equiv K(L,\beta;\ x,t)f_2(u) - K(0,\beta;\ x,t)f_1(u)$. Equation 3.10 may be considered as an integral equation for the determination of u.

We now wish to solve Equation 3.10. First we define $\bar{H}(x,t) \equiv H(u(x,t))$ and solve for $\bar{H}(x,t)$ by the classical method of successive approximations. We suppress the arguments α, β, x, and t for clarity and convenience. Also, as a matter of convenience, we arbitrarily pick as our initial approximation

$$H_0 = \int_0^L H(g)K\,d\alpha, \tag{3.11}$$

$$u_0 = H^{-1}(H_0). \tag{3.12}$$

We note that H^{-1} might not be single valued. However, we have the conditions that $F(u) > 0$ and that if H^{-1} is multiple valued it has a unique real, positive branch. These conditions establish the required branch of H^{-1}. We note also that if the branches of H^{-1} are not unique but are equivalent (for example, inverse trigonometric functions) the proof is valid.

In general we define

$$\begin{aligned} H_m \equiv H_0 &+ \int_0^t [Kf(u_{m-1}) - u_{m-1}K_\alpha]_{(0,L)}\,d\beta \\ &- \int_0^t\int_0^L \{K_{\alpha\alpha}[H(u_{m-1}) - u_{m-1}] + KG(u_{m-1})\}\,d\alpha\,d\beta \end{aligned} \tag{3.13}$$

where

$$(3.14) \qquad u_m \equiv H^{-1}(H_m).$$

In order to show the convergence of the sequence $\{H_m\}$ we first compute a sequence of majorizing terms for the sequence $\{|H_m - H_{m-1}|\}$. We then use the first majorizing sequence to construct a majorizing sequence for $\{H_m\}$ and thus establish convergence. First consider

$$(3.15) \qquad |H_1 - H_0| = \Big|\int_0^t \big[Kf(u_0) - u_0 K_\alpha \big]_{(0,L)} d\beta$$

$$- \int_0^t \int_0^L \{K_{\alpha\alpha}\big[H(u_0) - u_0\big] + KG(u_0)\} d\alpha \, d\beta \Big|$$

$$\leq m_0$$

where m_0 is defined as the supremum of the right hand side of Equation 3.15 over all x, $0 < x < L$, and all t, $0 \leq t \leq T$. All of the integrals in 3.15 exist by Lemma 2; hence m_0 is defined.

Since H and H^{-1} are continuously differentiable there exist constants M_1 and M_2 such that

$$|H^{-1}(H') - H^{-1}(H'')| \leq M_1 |H' - H''|$$

and $|H(V') - H(V'')| \leq M_2 |V' - V''|$ for H' and H" in a bounded domain D_1 and V' and V" in a bounded domain D_2. Hence we may write

$$(3.16) \qquad \begin{aligned} |u_1 - u_0| &= |H^{-1}(H_1) - H^{-1}(H_0)| \\ &\leq M_1|H_1 - H_0| \\ &\leq M_0 \end{aligned}$$

where

$$(3.17) \qquad M_0 \equiv M_1 m_0$$

We may now use Equations 3.15 and 3.16 to find majorizing terms for $|H_m - H_{m-1}|$, $m = 2, 3, \cdots$. By the hypotheses of the theorem and the above Lipschitz inequalities on H^{-1} and H we have

$$(3.18) \quad \begin{aligned} |H_2 - H_1| \leq & \left|\int_0^t \{K[f(u_1) - f(u_0)] - (u_1-u_0)K_\alpha\}_{(0,L)} d\beta\right| \\ & + \left|\int_0^t\int_0^L \{K_{\alpha\alpha}[H_1 - H_0 - u_1 + u_0] \right. \\ & \left. + K[G(u_1) - G(u_0)]\} d\alpha\, d\beta\right| \\ \leq & M_4\int_0^t K|u_1 - u_0|_{(0,L)} d\beta \\ & + \int_0^t |K_\alpha|\,|u_1 - u_0|_{(0,L)} d\beta \\ & + \int_0^t\int_0^L |K_{\alpha\alpha}|\,|u_1 - u_0|(1 + M_2) d\alpha\, d\beta \\ & + M_3\int_0^t\int_0^L K|u_1 - u_0| d\alpha\, d\beta. \end{aligned}$$

By the mean value theorem for multiple integrals (See, for example, Apostol, (1).) there exist constants M_5, M_6, M_7, and

M_8, $0 \le M_j < \infty$, $j = 5, 6, 7, 8$, such that

$$(3.19) \qquad \int_0^t \int_0^L |K_{\alpha\alpha}| \; |u_1 - u_0| d\alpha \, d\beta \le \frac{M_5}{L} \int_0^t \int_0^L |u_1 - u_0| d\alpha \, d\beta$$

$$\le M_5 M_0 \int_0^t d\beta, \quad \text{(by 3.16)},$$

$$\le M_5 M_0 t$$

$$\le M_5 M_0 T.$$

In a similar manner

$$(3.20) \qquad \int_0^t \int_0^L K |u_1 - u_0| d\alpha \, d\beta \le M_6 M_0 T,$$

$$(3.21) \qquad \int_0^t K |u_1 - u_0|_{(0,L)} d\beta \le M_7 M_0 T,$$

and

$$(3.22) \qquad \int_0^t |K_\alpha| |u_1 - u_0|_{(0,L)} d\beta \le M_8 M_0 T.$$

If we substitute these results into Equation 3.18 we get

$$(3.23) \qquad |H_2 - H_1| \le M_0 \left[M_4 M_7 + M_8 + M_5(1 + M_2) + M_3 M_6 \right] T.$$

If we now define

$$(3.24) \qquad M \equiv M_4 M_7 + M_8 + M_5(1 + M_2) + M_3 M_6$$

we may write Equation 3.23 in the form

$$(3.25) \qquad |H_2 - H_1| \le M_0 M T.$$

Hence we may use Equation 3.25 to write

$$(3.26) \qquad |u_2 - u_1| = |H^{-1}(H_2) - H^{-1}(H_1)|$$
$$\leq M_1 |H_2 - H_1|$$
$$\leq M_0 M_1 MT.$$

If we continue in this manner we may use induction to write, in general,

$$(3.27) \qquad |H_m - H_{m-1}| \leq m_0 (M_1 M)^{m-1} T^{m-1}.$$

Hence we have majorized the sequence $\{|H_m - H_{m-1}|\}$ by the sequence $\{m_0 (M_1 M)^{m-1} T^{m-1}\}$. We now use this result to majorize the sequence $\{H_m\}$.

First consider

$$(3.28) \qquad H_0 + [H_1 - H_0] + [H_2 - H_1] + \cdots + [H_m - H_{m-1}] = H_m.$$

By the triangle inequality we may write

$$(3.29) \qquad |H_m| \leq |H_0| + |H_1 - H_0| + \cdots + |H_m - H_{m-1}|$$
$$\leq m_0 (1 + M'T + M'^2 T^2 + \cdots + M'^{m-1} T^{m-1}) + |H_0|$$
$$\leq \frac{m_0}{1 - M'T} + |H_0|, \text{ for } M'T < 1, \text{ where } M' = MM_1.$$

Hence the series, whose m^{th} partial sum is Equation 3.28, has the geometric series as a majorant. By definition, the H_m are continuous. Hence for T chosen such that $M'T < 1$ and for $(x,t) \epsilon D(T)$, Equation 3.28 converges uniformly to a continuous function $\bar{H}(x,t)$ and

$$(3.30)\qquad \bar{H}(x,t) = \lim_{m\to\infty} H[u_m(x,t)] = H[\lim_{m\to\infty} u_m(x,t)]$$

since H is continuous. Since the above limit exists and H^{-1} exists and is continuous, the sequence $u_m(x,t)$ converges uniformly. Define $u(x,t) \equiv H^{-1}[\bar{H}(x,t)] = \lim_{m\to\infty} u_m(x,t)$. It remains to show that $u(x,t)$ satisfies Equation 3.10. To this end we consider

$$(3.31)\qquad \begin{aligned} H(u) &= \lim_{m\to\infty} H(u_m) \\ &= \lim_{m\to\infty} \Big\{ H_0 + \int_0^t [Kf(u_{m-1}) - u_{m-1}K_\alpha]_{(0,L)} d\beta \\ &\quad - \int_0^t\int_0^L \{K_{\alpha\alpha}[H(u_{m-1}) - u_{m-1}] + KG(u_{m-1})\} d\alpha\, d\beta \Big\} \\ &= H_0 + \lim_{m\to\infty} \int_0^t [Kf(u_{m-1}) - u_{m-1}K_\alpha]_{(0,L)} d\beta \\ &\quad - \lim_{m\to\infty} \int_0^t\int_0^L \{K_{\alpha\alpha}[H(u_{m-1}) - u_{m-1}] \\ &\quad + KG(u_{m-1})\} d\alpha\, d\beta. \end{aligned}$$

But f, H, and G are continuous and the sequence $\{u_m\}$ converges uniformly to $u(x,t)$. Therefore,

$$(3.32)\qquad \begin{aligned} H(u) &= H_0 + \int_0^t [Kf(u) - uK_\alpha]_{(0,L)} d\beta \\ &\quad - \int_0^t\int_0^L \{K_{\alpha\alpha}[H(u) - u] + G(u)\} d\alpha\, d\beta \end{aligned}$$

and we see that u is a solution of Equation 3.10.

We now wish to establish that u(x,t) is also a solution of our original system 3.1 and 3.2. If we add and subtract the term $\int_0^t [Ku_\alpha]_{(0,L)} d\beta$ in Equation 3.10 we may write 3.10 in the form

$$(3.33) \qquad H(u) = \int_0^L H(g)K d\alpha + \int_0^t [Ku_\alpha - uK_\alpha]_{(0,L)} d\beta$$
$$- \int_0^t \int_0^L \{K_{\alpha\alpha}[H(u) - u] + G(u)\} d\alpha \, d\beta$$
$$+ \int_0^t [Kf(u) - Ku_\alpha]_{(0,L)} d\beta$$

We first note that

$$\lim_{t \to 0^+} H(u) = \lim_{t \to 0^+} \int_0^L H(g(\alpha))K(\alpha,0;\ x,t)d\alpha$$
$$= H(g(x))$$

by Lemma 2a. Since H is continuous we see, there, that $\lim_{t \to 0^+} u(x,t) = g(x)$, that is, u(x,t) satisfies the initial condition in Equation 3.2. We now recall that

$$H(u) = \lim_{y \to t^-} \int_0^L H(u(\alpha,y))K(\alpha,y;\ x,t)d\alpha$$

so that we may write

$$(3.34)\qquad H(u) - \int_0^L H(g(\alpha))K(\alpha,0;\ x,t)\,d\alpha$$

$$= \lim_{y \to t^-} \left\{ \int_0^L H(u(\alpha,y))K(\alpha,y;\ x,t)\,d\alpha - \int_0^L H(u(\alpha,0))K(\alpha,0;\ x,t)\,d\alpha \right\}$$

$$= \lim_{y \to t^-} \int_0^y \int_0^L \frac{\partial}{\partial\beta}\left[H(u(\alpha,\beta))K(\alpha,\beta;\ x,t) \right] d\alpha\, d\beta$$

$$= \int_0^t \int_0^L \frac{\partial}{\partial\beta}\left[H(u(\alpha,\beta))K(\alpha,\beta;\ x,t) \right] d\alpha\, d\beta.$$

If we first substitute Equation 3.34 into Equation 3.33 and recognize that Equation 3.33 is Equation 3.9 with the added term $\int_0^t \left[Kf(u) - Ku_\alpha \right]_{(0,L)} d\beta$ we may reverse the steps by which we arrived at Equation 3.9 to arrive at an equation which corresponds to Equation 3.5, namely

$$(3.35)\qquad 0 = \int_0^t \int_0^L \left[Ku_{\alpha\alpha} - H(u)K_{\alpha\alpha} \right] d\alpha\, d\beta - \int_0^t \int_0^L \frac{\partial}{\partial\beta}\left[H(u)K \right] d\alpha\, d\beta - \int_0^t \int_0^L KG(u,\alpha,\beta)\,d\alpha\, d\beta + \int_0^t \left[Kf(u) - Ku_\alpha \right]_{(0,L)} d\beta.$$

We next note that $\frac{\partial}{\partial\beta}\left[H(u)K \right] = F(u)u_\beta K + H(u)K_\beta$ and we may rewrite Equation 3.35 in the form

(3.36) $$0 = \int_0^t \int_0^L \{K[u_{\alpha\alpha} - F(u)u_\beta - G(u,\alpha,\beta)] - H(u)[K_{\alpha\alpha} + K_\beta]\}d\alpha\, d\beta + \int_0^t [Kf(u) - Ku_\alpha]_{(0,L)} d\beta.$$

Since $K_{\alpha\alpha} + K_\beta = 0$, we may write Equation 3.36 in the form

(3.37) $$\int_0^t \int_0^L K[u_{\alpha\alpha} - F(u)u_\beta - G(u,\alpha,\alpha)]d\alpha\, d\beta = \int_0^t [Ku_\alpha - Kf(u)]_{(0,L)} d\beta.$$

We now apply Lemma 3 to Equation 3.37 and we see that

(3.38) $$u_{xx} - F(u)u_t - G(u,x,t) = 0$$

(3.39) $$u_x(0,t) = f_1(u,t) \qquad u_x(L,t) = f_2(u,t).$$

Hence u(x,t) is a solution to the system 3.1 and 3.2 and our proof is completed. Uniqueness is discussed in the Appendix.

AN EXAMPLE

We have examined the application of the successive approximation approach to the solution of a one dimensional nonlinear heat conduction problem. Specifically we considered the system

$$
(4.1)\qquad
\begin{aligned}
u_{xx} - 2uu_t &= (u - 1)\exp(-t)\cos x,\\
u(x,0) &= \cos x,\\
u_x(0,t) &= 0,\\
u_x(\pi/2,t) &= -e^{-t},
\end{aligned}
$$

which has the known solution $u(x,t) = \exp(-t)\cos x$. The numerical computation of the double integral term in the corresponding derived integral equation proved to be an extremely difficult problem, at least by the standard numerical techniques with which we are familiar. We found that the truncation error introduced in our integration over a region containing the singularity at $(\alpha,\beta) = (x,t)$ is of such magnitude that the resulting computations would be quite unreliable. It is possible that some limit approaching technique could be applied to a refined numerical integration scheme to approximate these integrals. Based on our knowledge of current numerical techniques and computer capabilities, however, at the present time it seems likely that this successive approximation approach will prove to be quite difficult to compute numerically for nonlinear or nonhomogeneous problems in which the double terms arise.

On the other hand, we have applied this technique to a linear, homogeneous equation with nonlinear boundary conditions with reasonably good results. In this case the multiple integral terms vanish and one can use standard Simpson's Rule numerical integration on the remaining integrals. In particular we considered a problem of Stefan-Boltzman radiation from the end of a finite rod. This problem is described by the system

$$
\begin{aligned}
u_{xx} - u_t &= 0, \\
u(x,0^+) &= 1, \\
u_x(0,t) &= -1, \\
u_x(1,t) &= 1 - u^4(1,t).
\end{aligned}
\tag{4.2}
$$

The corresponding integral equation is

$$
\begin{aligned}
u(x,t) = \frac{1}{2\pi^{1/2}} \Bigg\{ \int_0^1 t^{-1/2} \exp \frac{-(x-\alpha)^2}{4t}\, d\alpha \\
+ \int_0^t \Bigg\{ \left[\frac{1-u^4(1,t)}{(t-\beta)^{1/2}} + \frac{u(1,t)(1-x)}{2(t-\beta)^{3/2}} \right] \cdot \\
\exp\left[\frac{-(1-x)^2}{4(t-\beta)} \right] \\
+ \left[\frac{u(0,t)x}{2(t-\beta)^{3/2}} + \frac{1}{(t-\beta)^{3/2}} \right] \exp\left[\frac{-x^2}{4(t-\beta)} \right] \Bigg\} d\beta \Bigg\}.
\end{aligned}
\tag{4.3}
$$

As our first approximation we used the initial condition $u(x,0) = 1$. We felt that this would provide adequate convergence properties for small values of t. The succeeding iterations were carried out on an IBM 7070 computer. The cost of computer time limited us to two iterations per point.

The numerical results for the second iteration $u_2(x,t)$ are presented in the table below. For purposes of comparison, the results of a linear approximation are also presented in the table. This linear approximation is based on the solution of the system

$$
\begin{aligned}
u_{xx} - u_t &= 0 \\
u(x,0) &= K_1 \\
u_x(0,t) &= K_2 \\
u_x(1,t) &= -K_3(u(1,t)-K_1)
\end{aligned}
\tag{4.4}
$$

which is known in analytic form [see Carslaw and Jaeger (3), p. 125]. The solution of Equation 4.4 is varied by changing K_1 and K_3 so that $u_x(1,t)$ approximates $1 - u^4(1,t)$. This technique has proved quite accurate when compared with experimental results. The approximate solution by this method is designed by $U(x,t)$ in the table. Generally, we are interested in the solution of the Stefan-Bolteman problem only at the radiating endpoint. Hence our numerical approximations are given for $(x,t) = (1,t)$ and $(x,t) = (0,t)$.

Table 1. Numerical results

t	$u_2(0,t)$	$U(0,t)$	$u_2(1,t)$	$U(1,t)$
0	1.000	1.000	1.000	1.000
0.02	1.160	1.160	1.000	1.000
0.05	1.252	1.249	1.000	1.001
0.1	1.361	1.354	1.003	1.006
1.0	2.074	2.036	1.276	1.155

We note that for smaller values of t the agreement between $u_2(x,t)$ and $U(x,t)$ is quite close. For larger values of t it appears that a better initial approximation must be used to increase the rate of convergence. Chambré, (4), describes in detail a method for choosing the initial approximation to enhance the rate of convergence.

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APPENDIX

Clearly, for the successive approximation of the solution of the system 3.1 and 3.2 to be of any practical use, we must have that the solution is unique. If the solution were not unique, for example, the particular solution represented by a given successive approximation might be completely dependent upon the initial approximation. Moreover, we would have no guarantee that a given successive approximation would not oscillate among the various possible solutions as we vary x and t.

Graffi (11), has provided a uniqueness proof for a system of the form 3.1 and 3.2 with $G(u,x,t) \equiv 0$. We need only to add the condition that f_1 and $-f_2$ are monotone non-decreasing and either f_1 or $-f_2$ (or both) is strictly increasing for our solution to be unique in the case of $G(u,x,t) \equiv 0$. Hence, at least for this case, our successive approximation may, in theory, be used to construct solutions to our problem.